

ON THE NODAL LINE OF A SECOND EIGENFUNCTION OF THE LAPLACIAN-DIRICHLET IN SOME ANNULAR DOMAINS WITH DIHEDRAL SYMMETRY

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ABSTRACT. Let Ω be a bounded annular $C^{1,1}$ domain in \mathbb{R}^2 which is left invariant under the action of the dihedral group D_n of isometries of \mathbb{R}^2 . We show that the nodal line of a second Dirichlet eigenfunction must intersect the boundary of Ω , under suitable conditions on $\frac{\partial}{\partial \theta}$.

1. INTRODUCTION

In 1967, L. Payne [11, 12] conjectured that a second eigenfunction of the Laplacian with Dirichlet boundary conditions cannot have a closed interior nodal curve for any bounded domain $\Omega \subset \mathbb{R}^2$, i.e. if u_2 is a solution of the problem [6, p. 6]

$$(1.1) \quad \begin{cases} -\Delta u_2 = \lambda_2 u_2 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

where λ_2 is the second Dirichlet eigenvalue of Ω , and the nodal line of u_2 is

$$N = \overline{\{x \in \Omega \mid u_2(x) = 0\}},$$

then we must have

$$(1.2) \quad N \cap \partial\Omega \neq \emptyset.$$

L. Payne [12] gave an explicit proof of this for domains with smooth boundary which are convex in x and symmetric about the y -axis. In 1987, C.-S. Lin [9] showed that it holds when Ω is symmetric under a rotation with angle $2\pi p/q$ where p, q are positive integers. It has since been established [1, 8, 10, 14, 15] that (1.2) holds true for all bounded, convex domains in \mathbb{R}^2 as well as for some simply-connected concave domains.

However, a characterization of all planar domains for which (1.2) holds is now an open question, as counterexamples have been found [7] within the class of non simply-connected domains.

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In this paper we show that (1.2) holds within a class of annular, dihedrally symmetric domains. The argument is an extension of L. Payne's proof [12].

Let (r, θ) denote polar coordinates in \mathbb{R}^2 centered at the origin $(0, 0)$ with $r \geq 0, \theta \in]-\pi, \pi]$. Our main result is the following:

Theorem 1.1. *Let Ω be an annular domain with $C^{1,1}$ -boundary which is left invariant under the action of the dihedral group D_n of isometries of \mathbb{R}^2 generated by the rotation ρ_n of \mathbb{R}^2 about some fixed point on the x -axis by an angle of $2\pi/n$, and the reflection σ of \mathbb{R}^2 about the x -axis. Assume that Ω is contained in the region $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and that the point $(0, 0) \notin \overline{\Omega}$. Let $\partial\Omega = C_0 \cup C_1$ where the Jordan curves C_0 and C_1 are the outer and inner boundaries of Ω respectively. Suppose further that the following conditions hold:*

- (1) $\frac{\partial}{\partial\theta}$ points outward from Ω , i.e. $\langle \frac{\partial}{\partial\theta}(p), \nu(p) \rangle > 0, \forall p \in C_1 \cap \{(r, \theta) \mid \theta < 0\}$,
(where $\nu(p)$ denotes the outward unit normal at p).
- (2) $\frac{\partial}{\partial\theta}$ points inward into Ω , i.e. $\langle \frac{\partial}{\partial\theta}(p), \nu(p) \rangle < 0, \forall p \in C_0 \cap \{(r, \theta) \mid \theta < 0\}$.

Then (1.2) holds, i.e. if $N = \overline{\{x \in \Omega \mid u_2(x) = 0\}}$ is the nodal line of a second eigenfunction u_2 of Ω then $N \cap \partial\Omega \neq \emptyset$.

Possible extensions of Theorem 1.1 to a wider class of domains will be indicated in §2. An example of a domain satisfying the above conditions is illustrated below.

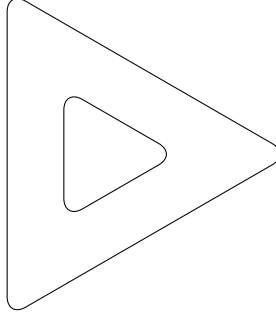


Figure 1.

2. PROOF OF THEOREM 1.1

If $u_2 \circ \sigma = -u_2$ there is nothing to show because the nodal line of u_2 is then the intersection of Ω with the x -axis.

Next consider the case where $u_2 \circ \sigma = u_2$.

Suppose if possible that $N \subset \Omega$.

By the Courant nodal domain theorem [4], $\Omega \setminus N$ has two connected components, D_- and D_+ , such that u_2 is strictly negative in D_- and strictly positive in D_+ .

Therefore as mentioned in [12, §2] and as a consequence of [3, Theorem 2.5], N must be a loop. Hence the following cases arise.

Case (i): $\partial D_- = N \cup C_1$ and $\partial D_+ = N \cup C_0$.

Were $\frac{\partial u_2}{\partial \theta} \geq 0$ below the x -axis, it would imply $u_2 \geq 0$ below the x -axis and hence in all of Ω , as $u_2 = u_2 \circ \sigma$. Since this is not possible the set $R = \{(r, \theta) \in \Omega \mid \theta < 0, \frac{\partial u_2}{\partial \theta}(r, \theta) < 0\}$ must be non-empty.

Now $\Delta u_2 = -\lambda_2 u_2 \geq 0$ in D_- and $u_2 = 0$ on ∂D_- . Since Ω has $C^{1,1}$ -boundary it satisfies the interior sphere condition at all points on $\partial \Omega$ by [2, Theorem 1.0.9]. Hence by the Hopf lemma [13], $\frac{\partial u_2}{\partial \theta}(q) > 0, \forall q \in C_1 \cap \{(r, \theta) \mid \theta < 0\}$, given that $\frac{\partial}{\partial \theta}(q)$ is directed outwards from D_- at these points.

Similarly $\Delta u_2 \leq 0$ in D_+ implies $\frac{\partial u_2}{\partial \theta}(q) > 0$ for points $q \in C_0 \cap \{(r, \theta) \mid \theta < 0\}$. Hence $\partial R \cap \partial \Omega$ is contained in the x -axis. Therefore $\frac{\partial u_2}{\partial \theta} = 0$ on ∂R .

Since Ω has $C^{1,1}$ -boundary, $\frac{\partial u_2}{\partial \theta} \in H^1(\Omega)$ [6, Theorem 1.2.10], and hence $\frac{\partial u_2}{\partial \theta} \in H^1(R \cup \sigma(R))$. As $\overline{R} \cap \overline{\sigma(R)}$ is contained in the x -axis, $\frac{\partial u_2}{\partial \theta} \in H_0^1(R \cup \sigma(R))$. Now $-\Delta(\frac{\partial u_2}{\partial \theta}) = \lambda_2 \frac{\partial u_2}{\partial \theta}$ in Ω in the weak sense. Therefore

$$-\int_{R \cup \sigma(R)} \left(\frac{\partial u_2}{\partial \theta} \right) \Delta \varphi \, dx = \lambda_2 \int_{R \cup \sigma(R)} \left(\frac{\partial u_2}{\partial \theta} \right) \varphi \, dx, \forall \varphi \in C_0^\infty(R \cup \sigma(R)).$$

Since $C_0^\infty(R \cup \sigma(R))$ is dense in $H_0^1(R \cup \sigma(R))$ it follows as a consequence of the Green's identity that

$$\lambda_2 = \frac{\int_{R \cup \sigma(R)} \|\nabla \left(\frac{\partial u_2}{\partial \theta} \right)\|^2 \, dx}{\int_{R \cup \sigma(R)} \left(\frac{\partial u_2}{\partial \theta} \right)^2 \, dx}$$

Also, $\frac{\partial u_2}{\partial \theta}$ changes sign in $R \cup \sigma(R)$. Thus by the variational principle for Dirichlet eigenvalues $\lambda_2(R \cup \sigma(R)) < \lambda_2$ [6, Formulae 1.35, Remark 1.2.4].

However this contradicts the domain monotonicity of Dirichlet eigenvalues [6, §1.3.2]. The argument is analogous when $\partial D_- = N \cup C_0$ and $\partial D_+ = N \cup C_1$.

Case (ii): $\partial D_- = N$ and $\partial D_+ = \partial \Omega \cup N$. Let

$$w = \sum_{i=0}^{n-1} u_2 \circ \rho_n^i$$

and let

$$K = \bigcup_{i=0}^{n-1} \rho_n^i(\overline{D_-}).$$

Then $\overline{D_-} \subset K$ and $\rho_n : K \rightarrow K$ is an isometry. Therefore as u_2 is strictly positive in $\Omega \setminus K$, w is also strictly positive in $\Omega \setminus K$. Since w is non-zero, it is a second Dirichlet eigenfunction of Ω . The nodal line of w is contained in K and therefore

does not intersect the boundary $\partial\Omega$. Now

$$w \circ \sigma = \sum_{i=0}^{n-1} u_2 \circ \rho_n^i \circ \sigma = \sum_{i=0}^{n-1} u_2 \circ \sigma \circ \rho_n^{-i} = \sum_{i=0}^{n-1} u_2 \circ \rho_n^{-i} = w$$

Also $w(q) = 0 \implies w(\rho_n^i(q)) = 0, \forall i = 0, \dots, n-1$. Hence the nodal line of w must be a loop encircling C_1 .

Therefore the argument in case (i) applies to w and this leads to a contradiction. The situation is analogous when $\partial D_+ = N$ and $\partial D_- = \partial\Omega \cup N$.

Next we express u_2 as the sum $u_2 = v + w$ where $v = \frac{u_2 - u_2 \circ \sigma}{2}$ and $w = \frac{u_2 + u_2 \circ \sigma}{2}$. Suppose if possible that $N \subset \Omega$.

Since C_0 is a compact subset of $\overline{\Omega} \setminus N$, there exists an ϵ -neighbourhood

$$U_0 = \bigcup_{p \in C_0} B_\epsilon(p) \cap \overline{\Omega}$$

of C_0 in $\overline{\Omega} \setminus N$ for some $\epsilon > 0$.

Since $U_0 \setminus C_0$ is connected, u_2 carries one sign in $U_0 \setminus C_0$, say $u_2|_{U_0 \setminus C_0} > 0$. If $v \leq 0$ in $U_0 \cap \{(r, \theta) \mid \theta \leq 0\}$ then $w = u_2 - v \geq 0$ in $U_0 \cap \{(r, \theta) \mid \theta \leq 0\}$ and hence $w \geq 0$ in U_0 by symmetry. The same argument also works if $v \leq 0$ in $U_0 \cap \{(r, \theta) \mid \theta \geq 0\}$. As the nodal line of v is the x -axis only these two possibilities exist.

Thus if $q \in C_0$ is a point on the nodal curve of w then w does not change sign in $B_\epsilon(q) \cap \Omega$, which is a contradiction. Hence the nodal curve of w does not intersect C_0 . A similar argument shows that the nodal curve of w does not intersect C_1 . But this contradicts what was shown earlier, because $w = w \circ \sigma$ and w is also a second eigenfunction of Ω .

Therefore the nodal line of u_2 must intersect the boundary $\partial\Omega$. \square

Concluding Remarks:

- (1) The above proof will go through for a wider class of dihedrally symmetric annular domains provided we are able to show that $\frac{\partial u_2}{\partial \theta}$ is in $H_0^1(R \cup \sigma(R))$. This holds for instance, when the boundary of Ω is polygonal [5, p. IX] and the rest of the conditions of Theorem 1.1 are met. Conditions (1) and (2) of Theorem 1.1 and the interior sphere condition are only required to hold almost everywhere on $\partial\Omega$ but we leave these considerations as a topic for future work as of now.
- (2) The choice of the x -axis in Theorem 1.1 is a matter of convenience. The proof goes through if polar coordinates are chosen in a way that the polar axis is an axis of reflection of Ω and the pole is away from Ω .

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